

On Some Improvements of the Brun–Titchmarsh Theorem. IV

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The aim of the present note is to give a version of large sieve extensions of the Brun–Titchmarsh theorem. This is in fact a rework of our old file left unpublished since early 1980's which we originally intended to include into our Tata lecture notes [12]. We publish it here in a slightly reworked form, as it might have now some fresh interest in view of Maynard's recent publication [4].

1. Let

$$\pi(x; k, \ell) = \sum_{\substack{p \leq x \\ p \equiv \ell \pmod k}} 1, \quad (k, \ell) = 1, \quad (1.1)$$

where p denotes a generic prime, and let χ be a Dirichlet character. Then [12, Theorem 13] asserts, among other things, that we have, uniformly for $kQ^2 \leq x^{9/20-\varepsilon}$,

$$\sum_{\substack{q \leq Q \\ (q, k) = 1}} \sum_{\chi \pmod q}^* \left| \sum_{\substack{p \leq x \\ p \equiv \ell \pmod k}} \chi(p) \right|^2 \leq \frac{(2 + o(1))x}{\varphi(k) \log(x/(kQ^2)^{3/8})} \pi(x; k, \ell), \quad (1.2)$$

provided x is larger than a constant which is effectively computable for each $\varepsilon > 0$, where φ is the Euler totient function and \sum^* stands for the restriction of the sum to primitive characters. In particular we have

$$\pi(x; k, \ell) \leq \frac{(2 + o(1))x}{\varphi(k) \log(x/k^{3/8})}, \quad k \leq x^{9/20-\varepsilon}, \quad (1.3)$$

which surpasses partly the famed bound

$$\pi(x; k, \ell) \leq \frac{2x}{\varphi(k) \log(x/k)}, \quad k < x, \quad (1.4)$$

due to Montgomery and Vaughan [5]. In contrast to this, Maynard [4] asserts in essence that

$$\pi(x; k, \ell) \leq \frac{2x}{\varphi(k) \log x}, \quad k \leq x^{1/8}, \quad (1.5)$$

provided x is larger than an effectively computable constant. He gives also a lower bound, though we skip it in order to make our presentation simple; for the same reason, we also skip mentioning former improvements upon (1.4) other than (1.3). The bound (1.5) has been known as a kind of folklore among specialists, but with less precision about the range of moduli.

We shall refine (1.2) by

Theorem. *There exists an effectively computable constant ω such that we have, uniformly for $kQ^2 \leq x^\omega$,*

$$\sum_{\substack{q \leq Q \\ (q, k) = 1}} \sum_{\chi \pmod q}^* \left| \sum_{\substack{p \leq x \\ p \equiv \ell \pmod k}} \chi(p) \right|^2 \leq \frac{2x}{\varphi(k) \log x} \pi(x; k, \ell). \quad (1.6)$$

Obviously this contains (1.5) but for $k \leq x^\omega$. It remains thus to find a good lower bound for ω . We are certain that Maynard's argument will extend to the direction indicated by (1.6) and yield (1.5) as a particular instance, since the basic structure of his argument is essentially the same as ours that is developed in [11][12] in detail, although the intricate part of [4] corresponding to the numerical precision should be overhauled accordingly. Further, we add that it is very possible to prove a short interval version of (1.6).

REMARK: The web edition of [12] contains some obvious misprints; for example the statement there corresponding to (1.2) lacks the necessary restriction to primitive characters. We are going to provide Tata IFR with corrections.

2. Our theorem is in fact a simple consequence of our version [11][12, Theorem 17] of the Linnik–Fogels–Gallagher prime number theorem; thus we need first to introduce a notion concerning zeros of Dirichlet L -functions $L(s, \chi)$. Details can be found in [16], for instance.

We consider the set Z_T of all non-trivial zeros in the region $|\operatorname{Im} s| \leq T$ of the function $\prod_{q \leq T} \prod_{\chi \bmod q}^* L(s, \chi)$, i.e., with χ being primitive. Then we have that there exists an effectively computable absolute constant $a_0 > 0$ such that

$$\max_{\rho \in Z_T} \operatorname{Re} \rho \leq 1 - \frac{a_0}{\log T}, \text{ excepting a possibly existing zero } \beta. \quad (2.1)$$

If β exists, it is real and simple, and we describe both itself and the relevant unique primitive character as T -exceptional. We put

$$\Delta_T = \begin{cases} 1 & \text{if } \beta \text{ does not exist,} \\ (1 - \beta) \log T & \text{if } \beta \text{ exists.} \end{cases} \quad (2.2)$$

It is generally believed that β does not exist. A way to confirm this is to improve appropriately the Brun–Titchmarsh theorem, which remains, however, to be one of the most difficult problems in analytic number theory; see [9][12, §4.3].

We put

$$\psi(x, \chi) = \sum_{n \leq x} \chi(n) \Lambda(n), \quad (2.3)$$

with the von Mangoldt function Λ ; and let

$$\tilde{\psi}(x, \chi) = \begin{cases} \psi(x, \chi) - x & \text{if } \chi \text{ is principal,} \\ \psi(x, \chi) + x^\beta / \beta & \text{if } \chi \text{ is } T\text{-exceptional,} \\ \psi(x, \chi) & \text{otherwise.} \end{cases} \quad (2.4)$$

Then, [12, Theorem 17] asserts that there exist effectively computable absolute constants $a_1, a_2, a_3 > 0$ such that

$$\sum_{q \leq T} \sum_{\chi \bmod q}^* |\tilde{\psi}(x, \chi)| \leq a_1 x \Delta_T \exp(-a_2 \log x / \log T), \quad (2.5)$$

provided $T^{a_3} \leq x \leq \exp((\log T)^2)$.

In order to prove our theorem, we note that

$$\begin{aligned} \sum_{\substack{p \leq x \\ p \equiv \ell \pmod{k}}} \chi(n) \Lambda(n) &= \frac{1}{\varphi(k)} \sum_{\xi \pmod{k}} \bar{\xi}(\ell) \psi(x, \xi \chi) \\ &= \frac{1}{\varphi(k)} \sum_{\xi \pmod{k}} \bar{\xi}(\ell) \psi(x, \xi^\# \chi) + O(\nu(k) \log x), \end{aligned} \quad (2.6)$$

where $\xi^\#$ is the primitive character inducing the Dirichlet character ξ , and $\nu(k)$ the number of distinct prime factors of k . Here $\xi^\# \chi$ stands for a unique primitive character whose conductor is not larger than kQ . We have thus

$$\begin{aligned} &\sum_{\substack{q \leq Q \\ (q,k)=1}} \sum_{\chi \pmod{q}}^* \left| \sum_{\substack{p \leq x \\ p \equiv \ell \pmod{k}}} \chi(n) \Lambda(n) \right| \\ &\leq \frac{1}{\varphi(k)} \{x + x^\beta / \beta + E(x, kQ)\} + O(\nu(k) Q^2 \log x), \end{aligned} \quad (2.7)$$

where β is the kQ -exceptional zero if exists; and $E(x, kQ)$ is the left side of (2.5) for $T = kQ$. If β exists, then (2.5) implies that

$$\begin{aligned} x^{-1} (x^\beta / \beta + E(x, kQ)) &\leq \exp(-\Delta_T \log x / \log T) / (1 - \Delta_T / \log T) \\ &\quad + a_1 \Delta_T \exp(-a_2 \log x / \log T), \end{aligned} \quad (2.8)$$

provided $T^{a_3} \leq x \leq ((\log T)^2)$. The right side is

$$\begin{aligned} &\leq \exp(-a_3 \Delta_T) / (1 - \Delta_T / \log T) + a_1 \Delta_T \exp(-a_2 a_3) \\ &\leq \exp(-a_3 \Delta_T) + \Delta_T (1 / (2 \log T) + a_1 \exp(-a_2 a_3)) \\ &< 1 - \frac{1}{2} a_3 \Delta_T, \end{aligned} \quad (2.9)$$

as we may assume that $a_3 \Delta_T$ is small while a_3 is large. Hence we have proved that if β exists, then

$$\sum_{\substack{q \leq Q \\ (q,k)=1}} \sum_{\chi \pmod{q}}^* \left| \sum_{\substack{p \leq x \\ p \equiv \ell \pmod{k}}} \chi(n) \Lambda(n) \right| \leq 2 \frac{x}{\varphi(k)} \left(1 - \frac{1}{5} a_3 \Delta_T\right), \quad (2.10)$$

provided $a_3 \Delta_T$ is small and $x \geq T^c$ with a computable absolute constant $c > 0$. The case where the T -exceptional zero does not exist is analogous; in fact, simpler. The rest of the proof may be skipped.

3. Both the bounds (1.2) and (2.5) and thus (1.6) are sieve results; that is, they are proved using mainly sieve arguments, without the zero-density theory or the Deuring–Heilbronn phenomenon. The proof in [12] of the assertion (1.2) depends on Iwaniec’s work [2] on the bilinear structure in the error term of the combinatorial linear sieve; an alternative approach to his result itself can be found in [12] (see also [1]). Prior to [2], a bilinear structure in the error term of the Selberg sieve was observed in [6] and the first uniform improvement upon (1.4) was achieved; see [3]. Later the development [13] made it possible to prove (1.2) via the Selberg sieve as well (see [17] for a further development). On the other hand, the bound (2.5) depends on our large sieve extension [8] of the Selberg sieve that is devised in [7] via

the duality principle and the quasi-character derived from optimal Λ^2 -weights. This line of consideration yielded a new way [10] to discuss zero-free regions of the zeta-function; in fact, it gave an assertion [14] that appears beyond the reach of the convexity argument of Borel, Carathéodory and Landau. A historical account of the developments in the modern theory of sieve methods can be found in [15][16].

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